MATB44 Week 10 Notes

1. General Theory of Linear Eqns:

– An n^{th} order linear differential eqn has the form: $P_0(t)\dfrac{d^n y}{dt^n} + P_1(t)\dfrac{d^{n-1}y}{dt^{n-1}} + \dots + P_{n-1}(t)\dfrac{dy}{dt} + P_n(t)y = G(t)$.

– Since we will be dealing with 3^{rd} order linear differential eqns at most, here is the form for 3^{rd} order linear differential eqns: $P_0(t)y''' + P_1(t)y'' + P_2(t)y' + P_3(t)y = G(t)$.

Note: If $P_0(t) \neq 0$, we can divide both sides of the eqn by it. Then, we get $y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = g(t)$.

Note: If $G(t)$ or $g(t) = 0$, then the differential eqn is homogeneous. Otherwise, it is non-homogeneous.

– The existence and uniqueness thm states that given the initial conditions $y(t_0) = y_0$, $y'(t_0) = y_0'$, and $y''(t_0) = y_0''$ and assuming all functions p_k are continuous, there exists a unique soln $y(t)$ and it is defined everywhere the eqn is defined.

– Superposition of solns of homogeneous eqn:
If y_1, y_2, y_3 solve $y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0$, then for any constant coefficients c_k, the linear combination $y = c_1y_1 + c_2y_2 + c_3y_3$ also solves the eqn.

- The Wronksian for 3 solns is defined as

$$W[y_1, y_2, y_3] = \begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1'(t) & y_2'(t) & y_3'(t) \\ y_1''(t) & y_2''(t) & y_3''(t) \end{vmatrix}$$

- y_1, y_2 and y_3 is a fundamental set of solns iff $W[y_1, y_2, y_3] \neq 0$.

- To get Abel's Formula, we need to differentiate W.

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}'$$

$$= \underbrace{\begin{vmatrix} y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}}_{\text{Here, we are differentiating the first row.}} + \underbrace{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1'' & y_2'' & y_3'' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}}_{\text{Here, we are differentiating the second row.}} + \underbrace{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}}_{\text{Here, we are differentiating the 3rd row.}}$$

Notice that for the first 2 determinants, there are 2 rows that are the same. Hence, the rows are linearly dependent and as a result, the determinant equals 0. So, we are only left with the last determinant.

$$= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}$$

Recall that $y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0$.
Hence, $y''' = -p_1(t)y'' - p_2(t)y' - p_3(t)y$.
Sub y''' into the determinant above.

$$= -p_1 \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} - p_2 \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \end{vmatrix} - p_3 \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Equals 0
because the
rows are
linearly dependent.

$$= -p_1 \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$= -p_1 W$$

$$W' = -p_1 W$$

$$\frac{W'}{W} = -p_1(t)$$

$$\frac{1}{W}\frac{dW}{dt} = -p_1(t)$$

$$\frac{1}{W}\, dW = -p_1(t)\, dt$$

$(\ln(w))' = -p_1(t)\,dt$
$\int (\ln(w))'\,dw = \int -p_1(t)\,dt$
$\ln(w) + c = \int -p_1(t)\,dt$
$\ln(w) = c + \int -p_1(t)\,dt$

$w = e^{c + \int -p_1(t)\,dt}$
$= e^{c} \cdot e^{-\int p_1(t)\,dt}$
$= c' \cdot e^{-\int p_1(t)\,dt}$ ← Abel's Formula

Once again there is a dichotomy. Either $w=0$ for all t or $w \neq 0$ for all t. This is because either $c'=0$ or $c' \neq 0$ and the exponent never equals to 0.

- Let y_1, y_2, and y_3 be functions. y_1, y_2, and y_3 are linearly independent if $c_1y_1 + c_2y_2 + c_3y_3 \neq 0$ at at least 1 point unless $c_1 = c_2 = c_3 = 0$. This definition applies to any functions, not necessarily solns of a homogeneous eqn.

Another way to think about this is y_1, y_2, and y_3 are linearly independent iff $W[y_1, y_2, y_3] \neq 0$ at at least 1 point.

E.g. 1 Determine whether $y_1 = 2t-3$, $y_2 = t^2+1$ and $y_3 = 2t^2 - t$ are linearly independent.
Soln:

$$W = \begin{vmatrix} 2t-3 & t^2+1 & 2t^2-t \\ 2 & 2t & 4t-1 \\ 0 & 2 & 4 \end{vmatrix}$$

$$= (2t-3)\left[8t - 2(4t-1)\right] - (t^2+1)\left[8\right] + (2t^2-t)\left[4\right]$$
$$= (2t-3)(8t-8t+2) - 8t^2 - 8 + 8t^2 - 4t$$
$$= 4t - 6 - 8t^2 - 8 + 8t^2 - 4t$$
$$= -14$$
$$\neq 0$$

Hence, Y_1, Y_2, and Y_3 are linearly independent.

- Suppose we have $ay''' + by'' + cy' + dy = 0$.
We let $y = e^{rt}$.
Then, $y' = re^{rt}$, $y'' = r^2 e^{rt}$, and $y''' = r^3 e^{rt}$.
Now, we have $ar^3 e^{rt} + br^2 e^{rt} + cre^{rt} + de^{rt} = 0$.
$e^{rt}(ar^3 + br^2 + cr + d) = 0$
$ar^3 + br^2 + cr + d = 0$

E.g. 2 Find the general soln of $y''' - y'' - y' + y = 0$.
Soln:

$$r^3 - r^2 - r + 1 = 0$$
$$(r^3 - r^2) - (r-1) = 0$$
$$r^2(r-1) - (r-1) = 0$$
$$(r-1)(r^2-1) = 0$$
$$(r-1)(r-1)(r+1) = 0$$
$$r_1 = -1,\ r_2 = 1$$

Hence, the general soln is $c_1 e^{t} + c_2 e^{-t} + c_3 t e^{-t}$

E.g. 3 Find the general soln of $y''' - 3y'' + 3y' - y = 0$

Soln:

$r^3 - 3r^2 + 3r - 1 = 0$

$(r-1)^3 = 0$

The roots are $r_1 = 1$, $r_2 = 1$, $r_3 = 1$.

Hence, the general soln is $c_1e^t + c_2te^t + c_3t^2e^t$.

E.g. 4 Find the general soln of $y''' - 2y'' - y' + 2y = 0$

Soln:

$r^3 - 2r^2 - r + 2 = 0$

$(r^3 - 2r^2) - (r-2) = 0$

$r^2(r-2) - (r-2) = 0$

$(r-2)(r^2-1) = 0$

$(r-2)(r-1)(r+1) = 0$

The roots are $r_1 = 2$, $r_2 = 1$, $r_3 = -1$.

Hence, the general soln is $c_1e^{2t} + c_2e^t + c_3e^{-t}$.

E.g. 5 Find the general soln of $18y''' + 21y'' + 14y' + 4y = 0$

Soln:

$18r^3 + 21r^2 + 14r + 4 = 0$

$18r^3 + 9r^2 + 12r^2 + 6r + 8r + 4 = 0$

$(18r^3 + 9r^2) + (12r^2 + 6r) + (8r + 4) = 0$

$9r^2(2r+1) + 6r(2r+1) + 4(2r+1) = 0$

$(2r+1)(9r^2 + 6r + 4) = 0$

$r_1 = \frac{-1}{2}$

To factor $9r^2 + 6r + 4$, I'll use the quadratic formula.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-6 \pm \sqrt{36 - 144}}{18}$$

$$= \frac{-6 \pm \sqrt{-108}}{18}$$

$$= \frac{-6 \pm 6\sqrt{-3}}{18}$$

$$= \frac{-1 \pm \sqrt{3}i}{3}$$

$$r_2 = \frac{-1}{3} + \frac{\sqrt{3}i}{3}$$

$$\lambda = \frac{-1}{3}, \quad u = \frac{\sqrt{3}}{3}$$

$$e^{rt} = e^{\lambda t}(\cos(ut) + i\sin(ut))$$

$$= e^{-t/3}\left(\cos\left(\tfrac{\sqrt{3}}{3}t\right) + i\sin\left(\tfrac{\sqrt{3}}{3}t\right)\right)$$

$$= e^{-t/3}\left(\cos\left(\sqrt{3}/3\, t\right)\right) + e^{-t/3}\left(i\sin\left(\sqrt{3}/3\, t\right)\right)$$

Hence, the general soln is $C_1 e^{-t/2} +$ $C_2 e^{-t/3} \cos\left(\frac{\sqrt{3}}{3}t\right) + C_3 e^{-t/3} \sin\left(\frac{\sqrt{3}}{3}t\right)$.

E.g. 6 Find the soln for the initial value problem $y''' - y'' + y' - y = 0$ and $y(0) = 2$, $y'(0) = -1$, $y''(0) = -2$

Soln:

$$r^3 - r^2 + r - 1 = 0$$

$$r^2(r-1) + (r-1) = 0$$

$$(r-1)(r^2+1) = 0$$

$$r_1 = 1$$

$r^2+1=0$
$r^2=-1$
$r=\pm i$
$r_2=i \rightarrow \lambda=0,\ u=1$

$y_2=\cos(t),\ y_3=\sin(t)$

$y=C_1e^t+C_2\cos(t)+C_3(\sin t)$

$y(0)=2$
$2=C_1+C_2$

$y'(0)=-1$
$y'=C_1e^t-C_2\sin(t)+C_3\cos(t)$
$-1=C_1+C_3$

$y''(0)=-2$
$y''=C_1e^t-C_2\cos(t)-C_3\sin(t)$
$-2=C_1-C_2$

$C_1=0,\ C_2=2,\ C_3=-1$

Hence, the general soln is $y=2\cos t-\sin t$

E.g. 7 Find the general soln of $y'''-y=0$.
Soln:
$r^3-1=0$
We use the formula $(a^3-b^3)=(a-b)(a^2+ab+b^2)$ to factor r^3-1.
$(r-1)(r^2+r+1)=0$
$r_1=1$

We will use the quadratic eqn to factor r^2+r+1.

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$= \frac{-1 \pm \sqrt{1-4}}{2}$$
$$= \frac{-1 \pm \sqrt{-3}}{2}$$
$$= \frac{-1 \pm \sqrt{3}i}{2}$$

Take $-\frac{1}{2} + \frac{\sqrt{3}i}{2}$

$\lambda = \frac{-1}{2}, \quad u = \frac{\sqrt{3}}{2}$

$$y_2 = e^{\lambda t}\cos(ut)$$
$$= e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)$$

$$y_3 = e^{\lambda t}\sin(ut)$$
$$= e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

Hence, the general soln is $C_1e^t + C_2e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + C_3e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$.

Note: There are a few possible types of answers.

1. Real and Distinct roots. Here, $r_1 \neq r_2$, $r_1 \neq r_3$, and $r_2 \neq r_3$. In this case, the general soln is $C_1e^{r_1t} + C_2e^{r_2t} + C_3e^{r_3t}$. An example of this type is example 4 on page 6.

2. Repeated roots. There are 2 possibilities for this case:

   a) A root is repeated once. I.e. Say $r_1 = r_2$ but $r_1 \neq r_3$. Then, the general soln is $C_1 e^{r_1 t} + C_2 t e^{r_1 t} + C_3 e^{r_3 t}$. An example of this is example 2 on page 5.

   b) A root is repeated twice. I.e. $r_1 = r_2 = r_3$. Then, the general soln is $C_1 e^{r_1 t} + C_2 t e^{r_1 t} + C_3 t^2 e^{r_1 t}$. An example of this is example 3 on page 6.

3. Complex roots. Note that with complex roots, it may not be the case that all 3 roots are complex. An example of this is example 7 on page 8.

E.g. 8 Find if $y_1 = 2t-3$, $y_2 = 2t^2+1$, $y_3 = 3t^2+t$ are linearly dependent or not.

Soln:

$$\omega = \begin{vmatrix} 2t-3 & 2t^2+1 & 3t^2+t \\ 2 & 4t & 6t+1 \\ 0 & 4 & 6 \end{vmatrix}$$

$$= (2t-3)\left[24t - 4(6t+1)\right] - (2t^2+1)(+12) + (3t^2+t)(+8)$$
$$= (2t-3)(24t - 24t - 4) - 24t^2 - 12 + 24t^2 + 8t$$
$$= -8t + 12 - 24t^2 - 12 + 24t^2 + 8t$$
$$= 0$$

Hence, y_1, y_2, and y_3 are linearly dependent.

E.g. 9 Find the general soln of
$y''' + 5y'' + 6y' + 2y = 0$.
Soln:

$r^3 + 5r^2 + 6r + 2 = 0$
$r^3 + r^2 + 4r^2 + 4r + 2r + 2 = 0$
$r^2(r+1) + 4r(r+1) + 2(r+1) = 0$
$(r+1)(r^2+4r+2) = 0$
$r_1 = -1$

To factor r^2+4r+2, I'll use the quadratic formula.

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$= \frac{-4 \pm \sqrt{16-8}}{2}$$
$$= \frac{-4 \pm 2\sqrt{2}}{2}$$
$$= -2 \pm \sqrt{2}$$

$r_2 = -2+\sqrt{2}$, $r_3 = -2-\sqrt{2}$

$y_1 = e^{-t}$, $y_2 = e^{(-2+\sqrt{2})t}$, $y_3 = e^{(-2-\sqrt{2})t}$

Hence, the general soln is $C_1e^{-t} + C_2e^{(-2+\sqrt{2})t} + C_3e^{(-2-\sqrt{2})t}$

E.g. 10 Find the soln of the given initial value problem $y''' + y' = 0$, $y(0) = 0$, $y'(0) = 1$ and $y''(0) = 2$.

Soln:

$r^3 + r = 0$

$r(r^2+1) = 0$

$r_1 = 0 \rightarrow y_1 = 1$

$r^2 = -1$

$r = \pm i$

Use $r = i$

$\lambda = 0$, $u = 1$

$y_2 = \cos(t)$

$y_3 = \sin(t)$

Hence, the general soln is $y = C_1 + C_2\cos(t) + C_3\sin(t)$

$y(0) = 0$

$0 = C_1 + C_2$

$y'(0) = 1$

$y' = -C_2\sin(t) + C_3\cos(t)$

$1 = C_3$

$y''(0) = 2$

$y'' = -C_2\cos(t) - C_3\sin(t)$

$2 = -C_2$

$$\left.\begin{aligned} 0 &= C_1 + C_2 \\ 1 &= C_3 \\ 2 &= -C_2 \end{aligned}\right\} \quad \begin{aligned} C_1 &= 2 \\ C_2 &= -2 \\ C_3 &= 1 \end{aligned}$$

Hence, the answer is $2 - 2\cos t + \sin t$.

E.g. 11 Find the general soln to $4y''' + y' + 5y = 0$

Soln:

$4r^3 + r + 5 = 0$

-1 is a root to this eqn.

$4(-1)^3 - 1 + 5$

$= -4 - 1 + 5$

$= -5 + 5$

$= 0$

Hence, $r+1$ is a factor. We can do polynomial long division to find the other factors.

$$\begin{array}{r l}
 & 4r^2 - 4r + 5 \\
r+1 \,\big) & 4r^3 + 0r^2 + r + 5 \\
 & -(4r^3 + 4r^2) \\
 & \quad -4r^2 + r + 5 \\
 & \quad -(-4r^2 - 4r) \\
 & \qquad 5r + 5 \\
 & \qquad -(5r + 5) \\
 & \qquad\quad 0
\end{array}$$

$(r+1)(4r^2 - 4r + 5) = 4r^3 + r + 5$

To factor $4r^2 - 4r + 5$, I'll use the quadratic formula.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16 - 80}}{8}$$

$$= \frac{4 \pm 8i}{8}$$

$$= \frac{1}{2} \pm i \qquad \lambda = \frac{1}{2}, \ u = 1$$

$r_1 = -1$. Hence, $y_1 = e^{-t}$
Take $r = \frac{1}{2} + i$

$y_2 = e^{t/2}\cos(t)$
$y_3 = e^{t/2}\sin(t)$

Hence, the general soln is $C_1 e^{-t} + C_2 e^{t/2}\cos(t) + C_3 e^{t/2}\sin(t)$.

2. Review of Sum and Series Properties:

- $\sum_{i=0}^{n} a_i = a_0 + a_1 + \ldots + a_n$

- $\sum_{i=0}^{n} c a_i = c \sum_{i=0}^{n} a_i$, c is a constant

- $\sum_{i=0}^{n} (a_i \pm b_i) = \sum_{i=0}^{n} a_i \pm \sum_{i=0}^{n} b_i$

- $\sum_{i=s}^{n} a_i = \sum_{i=s+p}^{n+p} a_{i-p}$

E.g. Suppose we have the sum

$\sum_{i=1}^{5} i$. It is equal to $1+2+3+4+5$ or 15.

Now, suppose we shift the index by 2.
We now have the sum

$$\sum_{i=3}^{7} (i-2) = (3-2)+(4-2)+(5-2)+(6-2)+(7-2)$$
$$= 1+2+3+4+5$$
$$= 15$$

- $$\sum_{i=s}^{n} a_i = \sum_{i=s-p}^{n-p} a_{i+p}$$

Note: For most cases, s-p must be greater than or equal to 0.

E.g. Suppose we have the sum

$$\sum_{i=5}^{10} i = 5+6+7+8+9+10$$
$$= 45$$

Now, if we have the sum

$$\sum_{i=0}^{5} (i+5) = (0+5)+(1+5)+(2+5)+(3+5)+(4+5)+(5+5)$$
$$= 5+6+7+8+9+10$$
$$= 45$$

- $\sum_{i=0}^{n} a_i = \sum_{i=0}^{m} a_i + \sum_{i=m+1}^{n}$, where $0 \le m < n$

- $\sum_{i=a}^{n} a_i = \sum_{i=0}^{n} a_i - \sum_{i=0}^{a-1} a_i$

Note: This property is simply a rearrangement of the previous property.

- For this week of MATB44, we'll be using the power series. First, I will explain the ratio test and how it can help us see if a series converges or diverges.

- Roughly speaking, a series is a summation of numbers. Hence, we can represent series in summation notation.

- Ratio Test:

Suppose we have the following series

$\sum_{n=0}^{\infty} a_n$ and that for all n, $a_n \neq 0$.

The ratio test states that if:

$\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = 1$, the series may converge or diverge.

$\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} > 1$, the series diverges.

$\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} < 1$, the series converges.

Now, I'll talk about power series.

- Power Series:
- Has the general formula

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where a_n and x_0 are numbers.

- We can use the ratio test to find the radius of convergence.

$$\lim_{n \to \infty} \frac{|a_{n+1} (x - x_0)^{n+1}|}{|a_n (x - x_0)^n|} < 1$$

$$\lim_{n \to \infty} \frac{|a_{n+1} (x - x_0)|}{|a_n|} < 1$$

$$|x - x_0| \underbrace{\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}}_{\text{Call this } p} < 1$$

$$|x - x_0| p < 1$$

$$|x - x_0| < \frac{1}{p}$$

let $R = \frac{1}{p}$

$|x - x_0| < R$

R is called the radius of convergence.

Note: The power series will converge for $|x-x_0| < R$ and diverge for $|x-x_0| > R$.

E.g. 12 Find the radius of convergence for

$$\sum_{n=0}^{\infty} (-1)^n n (x-2)^n$$

Soln:

$$\lim_{n\to\infty} \frac{|(-1)^{n+1}(n+1)(x-2)^{n+1}|}{|(-1)^n (n)(x-2)^n|} < 1$$

$$|x-2| \lim_{n\to\infty} \frac{|(-1)^{n+1}(n+1)|}{|(-1)^n (n)|} < 1$$

$$|x-2| \underbrace{\lim_{n\to\infty} \frac{|n+1|}{|n|}}_{\text{Equals } 1} < 1$$

$|x-2| < 1$

This means that 1 is the radius of convergence.
Furthermore, $|x-2| < 1$

$$-1 < x-2 < 1$$

$$2-1 < x < 1+2$$

$$1 < x < 3$$

(1, 3) is the convergence interval/ convergence domain.

The convergence domain / convergence interval is the set of x for which the series converges. For power series, it is always $(x_0 - r, x_0 + r)$ where r is the radius of convergence.

E.g. 13 Find the radius of convergence and the convergence interval for

$$\sum_{n=1}^{\infty} \frac{(-3)^n (x-5)^n}{n 7^{n+1}}$$

Soln:

$$\lim_{n \to \infty} \frac{\left| \frac{(-3)^{n+1} (x-5)^{n+1}}{(n+1)(7^{n+2})} \right|}{\left| \frac{(-3)^n (x-5)^n}{n 7^{n+1}} \right|} < 1$$

$$|x-5| \lim_{n \to \infty} \frac{\left| \frac{(-3)^{n+1}}{(n+1)(7^{n+2})} \right|}{\left| \frac{(-3)^n}{n 7^{n+1}} \right|} < 1$$

$$|x-5| \lim_{n \to \infty} \left| \frac{(-3)^{n+1}}{(n+1)(7^{n+2})} \right| \left| \frac{n 7^{n+1}}{(-3)^n} \right| < 1$$

$$|x-5| \lim_{n \to \infty} \left| \frac{-3n}{7(n+1)} \right| < 1$$

$$\frac{3}{7} |x-5| \lim_{n \to \infty} \frac{|n|}{|n+1|} < 1$$

$$\lim_{n\to\infty} \frac{|n|}{|n+1|} = 1$$

$$\frac{3}{7}|x-5|<1$$

$$|x-5| < \frac{7}{3}$$

Hence, the radius of convergence is $\frac{7}{3}$

and the convergence interval is $(5-\frac{7}{3},\ 5+\frac{7}{3})$.

E.g. 14 Find the radius of convergence and the convergence interval for

$$\sum_{n=1}^{\infty} \frac{(-1)^n n (x+3)^n}{4^n}$$

Soln:

$$\lim_{n\to\infty} \left|\frac{(-1)^{n+1}(n+1)(x+3)^{n+1}}{4^{(n+1)}}\right| \left|\frac{4^n}{(-1)^n n (x+3)^n}\right| < 1$$

$$|x+3| \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(n+1)}{4^{(n+1)}}\right| \left|\frac{4^n}{(-1)^n n}\right| < 1$$

$$|x+3| \lim_{n\to\infty} \left|\frac{n+1}{4n}\right| < 1$$

$$\frac{|x+3|}{4} \lim_{n\to\infty} \left|\frac{n+1}{n}\right| < 1$$

$$\frac{|x+3|}{4} < 1$$

$$|x+3| < 4$$

$$-4 < x+3 < 4$$

$$-4-3 < x < 4-3$$

$$-7 < x < 1$$

Hence, the radius of convergence is 4 and the convergence interval is $(-7, 1)$.

E.g. 15 Find the radius of convergence and convergence interval for

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n \cdot 2^n}$$

Soln:

$$\lim_{n \to \infty} \left| \frac{(x+1)^{n+1}}{(n+1)2^{n+1}} \right| \left| \frac{n \cdot 2^n}{(x+1)^n} \right| < 1$$

$$|x+1| \lim_{n \to \infty} \left| \frac{n \cdot 2^n}{(n+1)2^{n+1}} \right| < 1$$

$$|x+1| \lim_{n \to \infty} \left| \frac{2n}{n+1} \right| < 1$$

$$\frac{|x+1|}{2} \lim_{n \to \infty} \left| \frac{n}{n+1} \right| < 1$$

$$\frac{|x+1|}{2} < 1 \rightarrow |x+1| < 2$$

Hence, the radius of convergence is 2 and the convergence interval is $(-3, 1)$.

3. Series Solns Near an Ordinary Point:

– Consider the differential eqn $p(x)y'' + q(x)y' + r(x)y = 0$, where p, q and r are nonconstant coefficients. We say that $x = x_0$ is an ordinary point if $p(x_0) \neq 0$ and a singular point if $p(x_0) = 0$.

E.g. 16 Find a series soln to $y'' - xy = 0$ near $x = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Hence, $y'' - xy$ can be written as

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Since we want both summations to have x^n, we need to shift the first sum down by 2 and shift the second sum up by 1.

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

Now, we will take out the $n=0$ term from the first sum.

$$(2)(1)a_2 x^0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} x^n\left((n+2)(n+1)a_{n+2} - a_{n-1}\right) = 0$$

Now, we will take various values for n to see if we can find a pattern.

Take $n=0$. Then, we have $2a_2 = 0 \Rightarrow a_2 = 0$.

Note: When we take $n=1, 2, \ldots$, we get

$(n+2)(n+1)a_{n+2} - a_{n-1} = 0$

$(n+2)(n+1)a_{n+2} = a_{n-1}$

$a_{n+2} = \dfrac{a_{n-1}}{(n+2)(n+1)}$

Take $n=1$.

$a_3 = \dfrac{a_0}{(3)(2)}$ ← Note: keep it like this. Don't multiply yet. We may need to cancel out terms.

Take $n=2$

$a_4 = \dfrac{a_1}{(4)(3)}$

Take $n=3$

$$a_5 = \frac{a_2}{(5)(4)}$$

$$= 0$$

Take $n=4$

$$a_6 = \frac{a_3}{(6)(5)}$$

$$= \frac{a_0}{(6)(5)(3)(2)}$$

We can see a recursive pattern. I will draw a chart to help see the pattern better.

$a_3 = \frac{a_0}{3.2}$	$a_4 = \frac{a_1}{4.3}$	$a_5 = \frac{a_2}{2 \cdot 1} = 0$
$a_6 = \frac{a_3}{6.5} = \frac{a_0}{6.5.3.2}$	$a_7 = \frac{a_4}{7.6} = \frac{a_1}{7.6.4.3}$	$a_8 = \frac{a_5}{8.7} = 0$
$\vdots$ $a_{3k} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot \ldots \cdot (3k-1)(3k)}$	$\vdots$ $a_{3k+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot \ldots \cdot (3k)(3k+1)}$	$\vdots$ $a_{3k+2} = 0$
$k = 1, 2, 3, \ldots$	$k = 1, 2, 3, \ldots$	$k = 0, 1, 2, 3, \ldots$

Recall that $y = \sum_{n=0}^{\infty} a_n x^n$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_{3k} x^{3k} + a_{3k+1} x^{3k+1} + \dots$$

Now, we can rewrite the eqn as

$$a_0 + a_1 x + \frac{a_0 x^3}{6} + \frac{a_1 x^4}{12} + \dots$$

$$y = a_0 \left(1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{2 \cdot 3 \cdot 5 \cdot 6 \cdot \dots \cdot (3k-1)(3k)}\right) +$$

$$a_1 \left(x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{3 \cdot 4 \cdot 6 \cdot 7 \cdot \dots \cdot (3k)(3k+1)}\right)$$

Note: a_0 and a_1 are arbitrary coefficients.

$$y_1 = 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{2 \cdot 3 \cdot 5 \cdot 6 \cdot \dots \cdot (3k-1)(3k)}$$

$$y_2 = x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{3 \cdot 4 \cdot 6 \cdot 7 \cdot \dots \cdot (3k)(3k+1)}$$

To see if y_1 and y_2 are a fundamental pair of solns, we will use the wronksian.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= y_1 y_2' - y_1' y_2$$

Since we need to show that $W \neq 0$ at at least 1 point, we will choose an easy point, 0.

$y_1(0) = 1$

$y_2(0) = 0$

Since we know $y_2(0) = 0$, we don't need to calculate $y_1'(0)$. It doesn't matter what it is.

$$y_2'(0) = \left(x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{3 \cdot 4 \cdot 6 \cdot 7 \cdot \ldots \cdot 3k \cdot 3k+1}\right)' \Big|_{x=0}$$

$$= \left(1 + \sum_{k=1}^{\infty} \frac{(3k+1)\, x^{3k}}{3 \cdot 4 \cdot 6 \cdot 7 \cdot \ldots \cdot 3k \cdot 3k+1}\right) \Big|_{x=0}$$

$$= 1$$

Hence, $W = 1 \cdot 1 - 0$

$= 1$

Hence, y_1 and y_2 are a fundamental pair of solns.